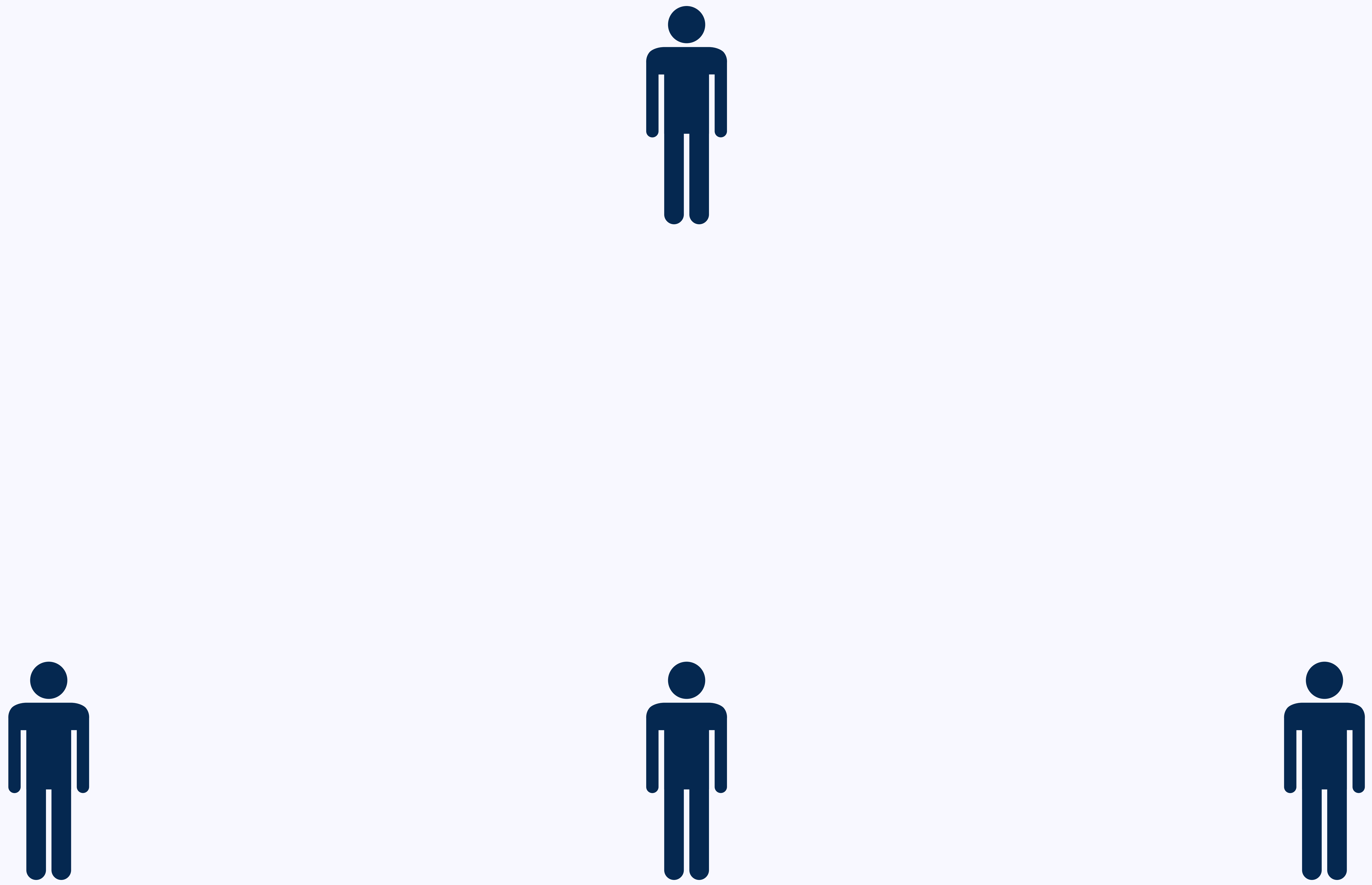
Rational Inattention and Endogenous Volatility: a Large Deviation Approach

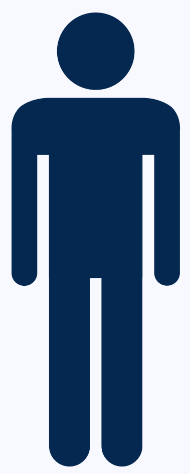
Tetsuya Hoshino Takashi Ui

BRIC 2025



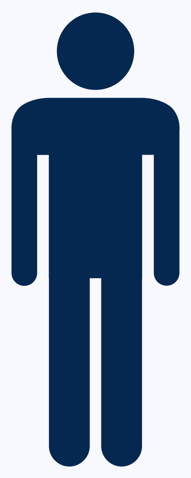
invest $(a_i = 1)$ or **not invest** $(a_i = 0)$

investment cost = state $\theta \sim$ prior π



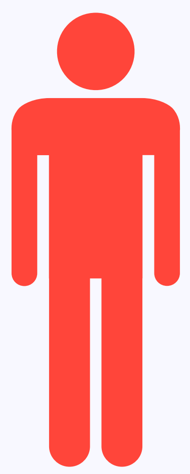
invest ($a_i = 1$) or **not invest** ($a_i = 0$)

investment cost = state $\theta \sim$ prior π



invest ($a_i = 1$) or **not invest** ($a_i = 0$)

investment cost = state $\theta \sim$ prior π

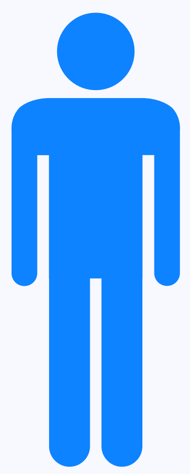


payoff from **investing** = average action $3/4 - \theta$

payoff from **not investing** $= 0$

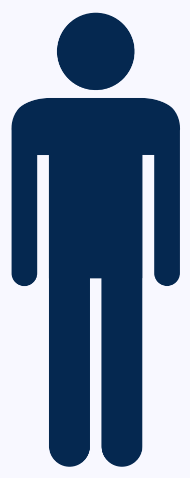
invest ($a_i = 1$) or **not invest** ($a_i = 0$)

investment cost = state $\theta \sim$ prior π



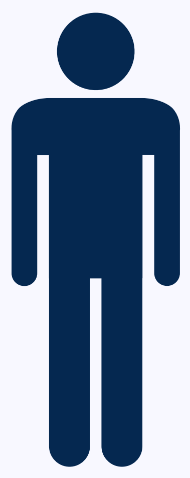
payoff from **investing** = average action $3/4 - \theta$

payoff from **not investing** $= 0$



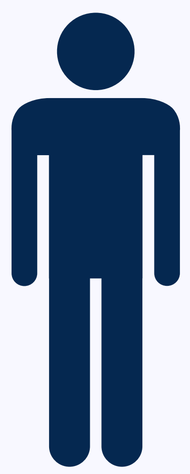
Agents are uncertain about...

an **exogenous** state θ and an **endogenous** average action $\bar{a}_n$



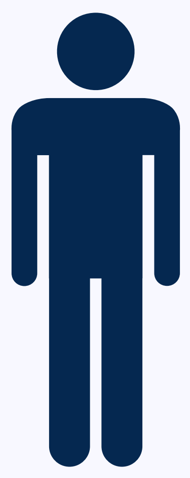
Agents may acquire information about…

an **exogenous** state θ and an **endogenous** average action $\bar{a}_n$



Equilibrium actions are **conditionally correlated** given state θ

The average action $\bar{a}_n = \frac{1}{n}\sum_i a_i$ has **endogenous volatility**



Equilibrium actions are **conditionally correlated** given state θ

The average action $\bar{a}_n = \frac{1}{n}\sum_i a_i$ has **endogenous volatility**


What if #agents → ∞?

rational inattention

How does **information acquisition** impact the equilibrium aggregate behavior in **large games**?

#agents $\rightarrow \infty$

rational inattention

How does **information acquisition** impact the equilibrium aggregate behavior in **large games**?

#agents $\to \infty$

rational inattention

How does **information acquisition** impact the equilibrium aggregate behavior in **large games**?

#agents $\to \infty$

Endogenous Volatility

exogenous volatility

variations of fundamentals

$$\text{Var}\left(\bar{a}_n\right) = \text{Var}\left(\mathbb{E}\left[\bar{a}_n \mid \theta\right]\right) + \mathbb{E}\left[\text{Var}\left(\bar{a}_n \mid \theta\right)\right]$$

endogenous volatility

endogenous info acquisition

endogenous correlation

$$\text{Var}\left(\bar{a}_n \mid \theta\right) = \frac{1}{n}\text{Var}\left(a_i \mid \theta\right) + \frac{n-1}{n}\text{Cov}\left(a_i, a_j \mid \theta\right)$$

Endogenous Volatility

exogenous volatility

variations of fundamentals

$$\text{Var}(\bar{a}_n) = \text{Var}(\mathbb{E}[\bar{a}_n \mid \theta]) + \mathbb{E}[\text{Var}(\bar{a}_n \mid \theta)]$$

endogenous volatility

endogenous info acquisition

endogenous correlation

$$\text{Var}(\bar{a}_n \mid \theta) = \frac{1}{n}\text{Var}(a_i \mid \theta) + \frac{n-1}{n}\text{Cov}(a_i, a_j \mid \theta)$$

Endogenous Volatility

exogenous volatility

variations of fundamentals

$$\mathrm{Var}\left(\bar{a}_n\right) = \mathrm{Var}\left(\mathbb{E}\left[\bar{a}_n \mid \theta\right]\right) + \mathbb{E}\left[\mathrm{Var}\left(\bar{a}_n \mid \theta\right)\right]$$

endogenous volatility

endogenous info acquisition

endogenous correlation

$$\mathrm{Var}\left(\bar{a}_n \mid \theta\right) = \frac{1}{n}\mathrm{Var}\left(a_i \mid \theta\right) + \frac{n-1}{n}\mathrm{Cov}\left(a_i, a_j \mid \theta\right)$$

Endogenous Volatility

exogenous volatility

variations of fundamentals

$$\mathrm{Var}\left(\bar{a}_n\right) = \mathrm{Var}\left(\mathbb{E}\left[\bar{a}_n \mid \theta\right]\right) + \mathbb{E}\left[\mathrm{Var}\left(\bar{a}_n \mid \theta\right)\right]$$

endogenous volatility

endogenous info acquisition

endogenous correlation

$$\mathrm{Var}\left(\bar{a}_n \mid \theta\right) = \frac{1}{n}\mathrm{Var}\left(a_i \mid \theta\right) + \frac{n-1}{n}\mathrm{Cov}\left(a_i, a_j \mid \theta\right)$$

Endogenous Volatility

exogenous volatility

variations of fundamentals

$$\mathrm{Var}\left(\bar{a}_n\right) = \mathrm{Var}\left(\mathbb{E}\left[\bar{a}_n \mid \theta\right]\right) + \mathbb{E}\left[\mathrm{Var}\left(\bar{a}_n \mid \theta\right)\right]$$

endogenous volatility

endogenous info acquisition

endogenous correlation

$$\mathrm{Var}\left(\bar{a}_n \mid \theta\right) \approx \quad \mathrm{Cov}\left(a_i, a_j \mid \theta\right)$$

Result Overview

Class of aggregative potential games

- e.g., beauty contests, bank runs, political revolution, voting

As #agents $n \to \infty$:

- **endogenous volatility vanishes:**

$$\text{Var}\big(\bar{a}_n \mid \theta\big) \approx \text{Cov}\big(a_i, a_j \mid \theta\big) \to 0$$

  - agents acquire info about state θ only

- **equilibrium** $=$ **representative agent's problem**

Result Overview

Class of aggregative potential games

- e.g., beauty contests, bank runs, political revolution, voting

As #agents $n \to \infty$:

- **endogenous volatility vanishes:**

$$\text{Var}\big(\bar{a}_n \mid \theta\big) \approx \text{Cov}\big(a_i, a_j \mid \theta\big) \to 0$$

  - agents acquire info about state θ only

- **equilibrium $=$ representative agent's problem**

Result Overview

Class of aggregative potential games

- e.g., beauty contests, bank runs, political revolution, voting

As #agents $n \to \infty$:

- **endogenous volatility vanishes:**

$$\text{Var}\left(\bar{a}_n \mid \theta\right) \approx \text{Cov}\left(a_i, a_j \mid \theta\right) \to 0$$

  - agents acquire info about state θ only

- **equilibrium $=$ representative agent's problem**

Result Overview

Class of aggregative potential games

- e.g., beauty contests, bank runs, political revolution, voting

As #agents $n \to \infty$:

- **endogenous volatility vanishes:**

$$\text{Var}\left(\bar{a}_n \mid \theta\right) \approx \text{Cov}\left(a_i, a_j \mid \theta\right) \to 0$$

  - agents acquire info about state θ only

- **equilibrium $=$ representative agent's problem**

Result Overview

Class of aggregative potential games

- e.g., beauty contests, bank runs, political revolution, voting

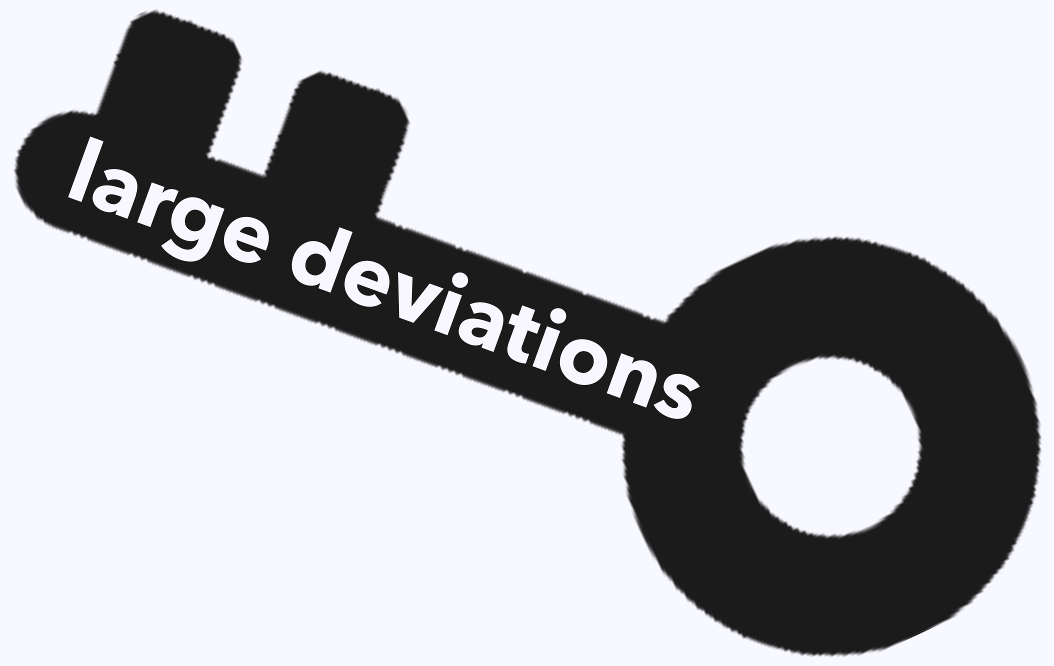
As #agents $n \to \infty$:

- **endogenous volatility vanishes:**

$$\text{Var}\big(\bar{a}_n \mid \theta\big) \approx \text{Cov}\big(a_i, a_j \mid \theta\big) \to 0$$

  - agents acquire info about state θ only

- **equilibrium $=$ representative agent's problem**



Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$
- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\,\bar{a}_n\right) + \frac{\bar{a}_n}{2}$
  - even if we replace u_i with v_n agents' behavior does not change

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n \to \infty} \frac{\bar{a}^2}{2} - \theta\,\bar{a}$$

Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$
- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\,\bar{a}_n\right) + \frac{\bar{a}_n}{2}$
  - even if we replace u_i with v_n agents' behavior does not change

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n \to \infty} \frac{\bar{a}^2}{2} - \theta\,\bar{a}$$

Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$
- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\,\bar{a}_n\right) + \frac{\bar{a}_n}{2}$
    - even if we replace u_i with v_n agents' behavior does not change

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n \to \infty} \frac{\bar{a}^2}{2} - \theta\,\bar{a}$$

Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$

- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\,\bar{a}_n\right) + \frac{\bar{a}_n}{2}$

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n\to\infty} \frac{\bar{a}^2}{2} - \theta\,\bar{a}$$

- even if we replace u_i with v_n agents' behavior does not change

Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n \to \infty} \frac{\bar{a}^2}{2} - \theta\, \bar{a}$$

- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\, \bar{a}_n\right) + \frac{\bar{a}_n}{2}$
  - even if we replace u_i with v_n agents' behavior does not change

Example

State θ

- common prior $\pi \in \Delta(\text{finite } \Theta)$

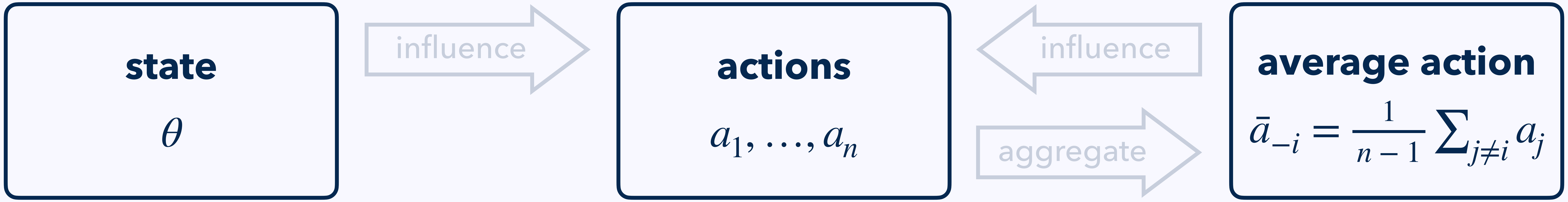
Agents $i = 1, 2, \ldots, n$

- action $a_i = 1$ or 0
- payoff $u_i(a, \theta) = \begin{cases} \bar{a}_n - \theta & \text{if } a_i = 1 \\ 0 & \text{if } a_i = 0 \end{cases}$
- potential $v_n(a, \theta) = n\left(\frac{\bar{a}_n^2}{2} - \theta\,\bar{a}_n\right) + \frac{\bar{a}_n}{2}$
  - even if we replace u_i with v_n agents' behavior does not change

strategic interaction ($n \to \infty$) is described by the **limit potential**:

$$\frac{v_n(a, \theta)}{n} \xrightarrow{n \to \infty} \frac{\bar{a}^2}{2} - \theta\,\bar{a}$$

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

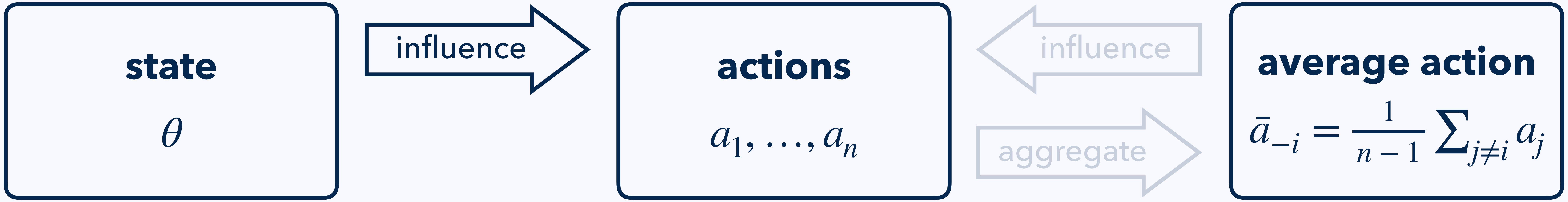
$$\max_{P_i} \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

Entropy cost

$$\text{info cost} = \text{unit cost } \lambda \times \text{uncertainty reduction in terms of entropy}$$

more

Rational Inattention Sims ’03 Matějka–McKay ’15 Caplin-Dean-Leahy ’21 Denti ’23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

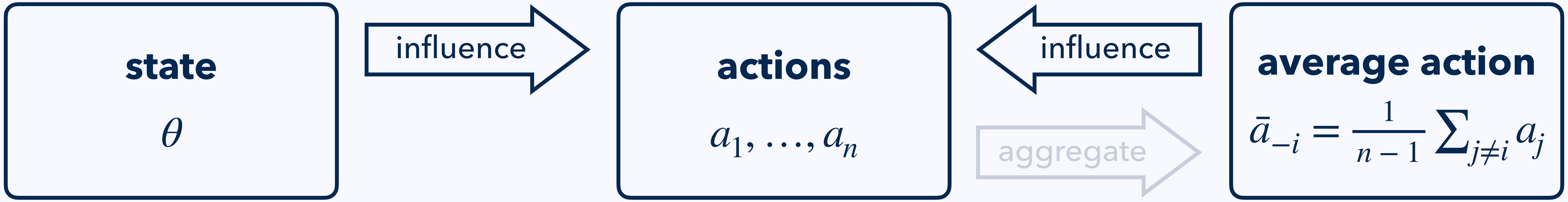
$$\max_{P_i} \ \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

Entropy cost

info cost $=$ unit cost λ $\times$ uncertainty reduction in terms of entropy

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

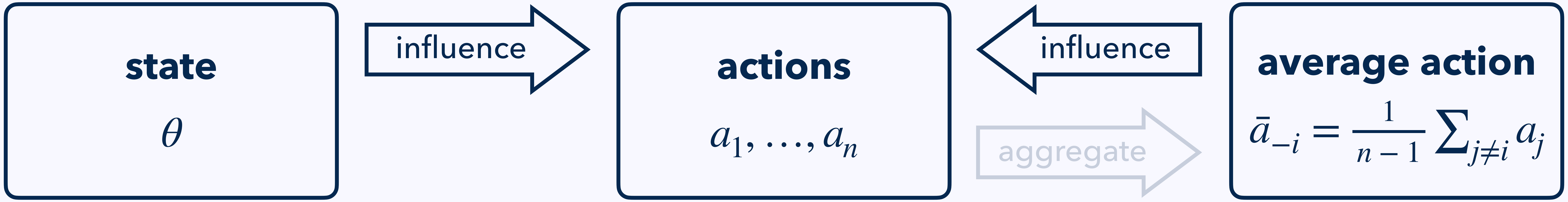
$$\max_{P_i} \; \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

Entropy cost

info cost $=$ unit cost λ $\times$ uncertainty reduction in terms of entropy

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

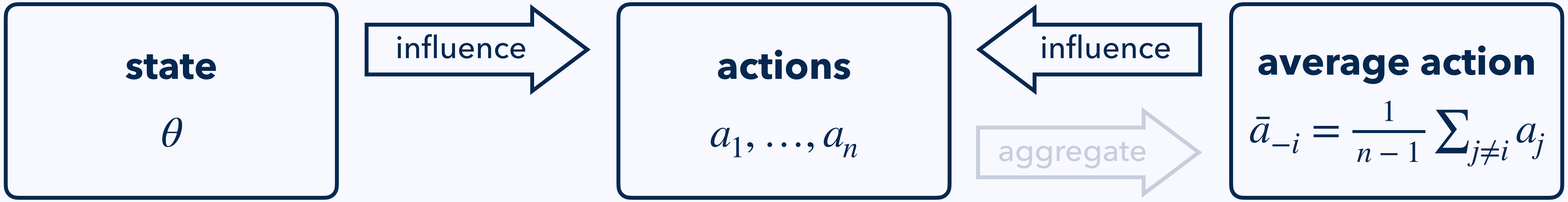
$$\max_{P_i} \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

Entropy cost

info cost = unit cost λ × uncertainty reduction in terms of entropy

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i\left(a_i \mid \bar{a}_{-i}, \theta\right)$:

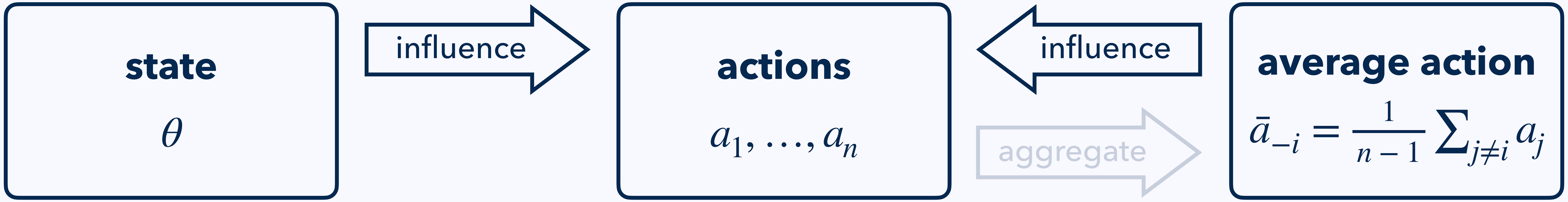
$$\max_{P_i} \; \mathbb{E}\left[u_i\left(a_i, a_{-i}, \theta\right)\right] - \text{info cost}$$

Entropy cost

$$\text{info cost} \;=\; \text{unit cost } \lambda \;\times\; \text{uncertainty reduction in terms of entropy}$$

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

$$\max_{P_i} \ \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

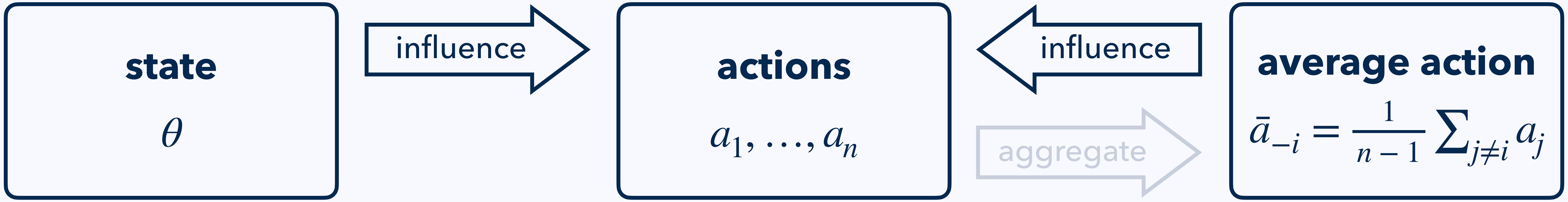
Entropy cost

$$\text{info cost} = \text{unit cost } \lambda \times \text{uncertainty reduction in terms of entropy}$$

$$= \text{prior entropy} - \mathbb{E}[\text{posterior entropy}]$$

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i(a_i \mid \bar{a}_{-i}, \theta)$:

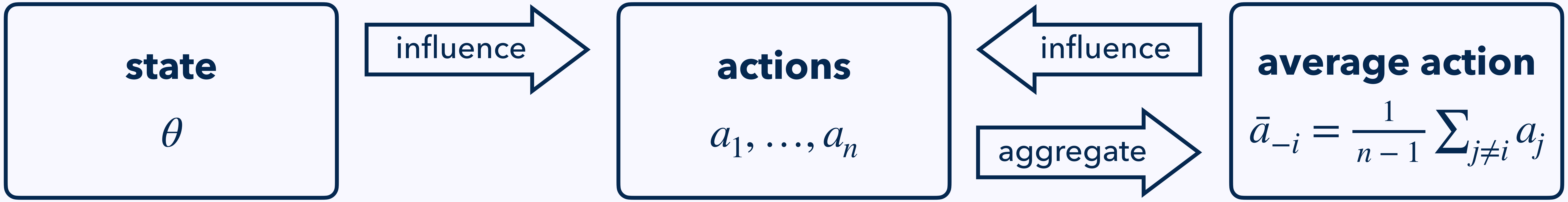
$$\max_{P_i} \; \mathbb{E}\left[u_i(a_i, a_{-i}, \theta)\right] - \text{info cost}$$

Entropy cost

$$\text{info cost} = \underbrace{\text{unit cost } \lambda}_{=1} \times \underbrace{\text{uncertainty reduction in terms of entropy}}_{= \text{prior entropy} - \mathbb{E}[\text{posterior entropy}]}$$

more

Rational Inattention Sims '03 Matějka–McKay '15 Caplin-Dean-Leahy '21 Denti '23



Agent i optimizes over conditional action distributions $P_i\left(a_i \mid \bar{a}_{-i}, \theta\right)$:

$$\max_{P_i} \; \mathbb{E}\left[u_i\left(a_i, a_{-i}, \theta\right)\right] - \text{info cost}$$

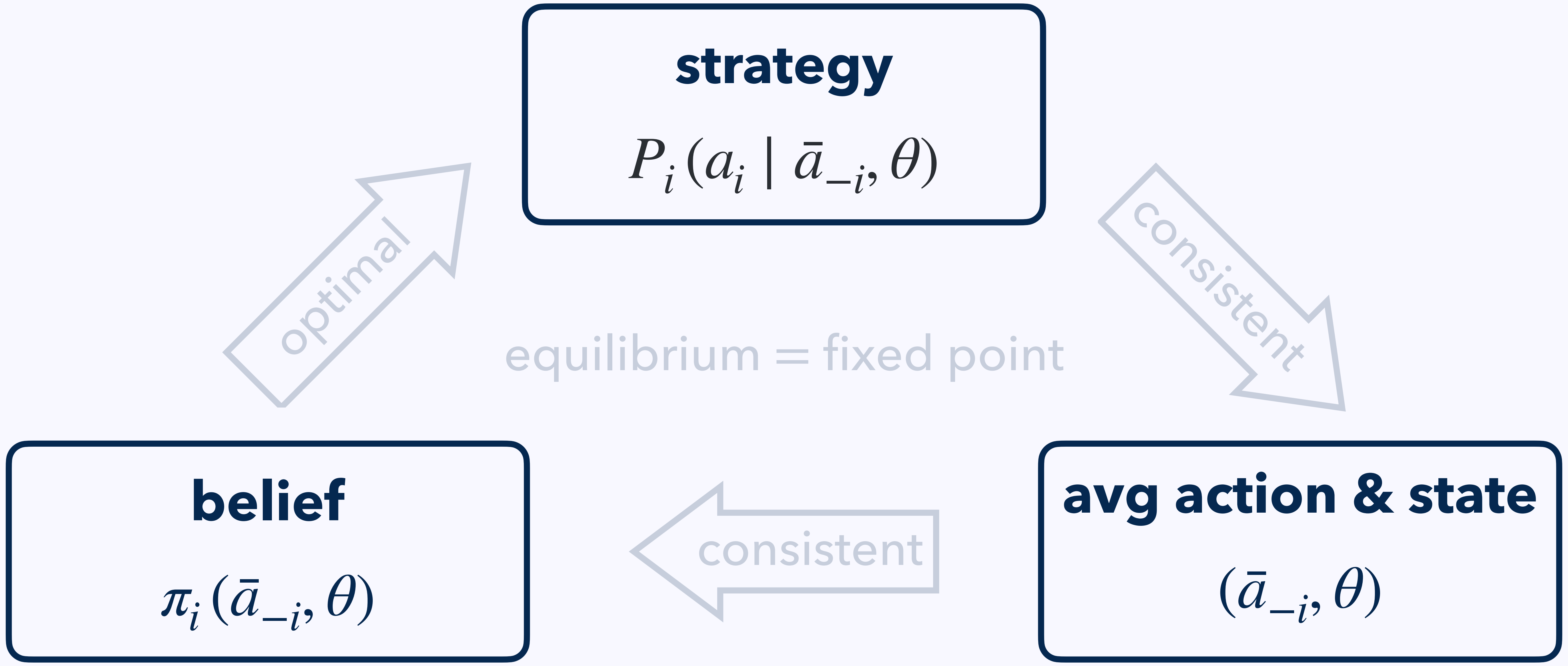
Entropy cost

info cost $=$ unit cost λ $\times$ uncertainty reduction in terms of entropy

$= 1$ $\qquad$ $=$ prior entropy $-$ $\mathbb{E}$[posterior entropy]

more

Equilibrium Denti '23 Hébert–La'O '23

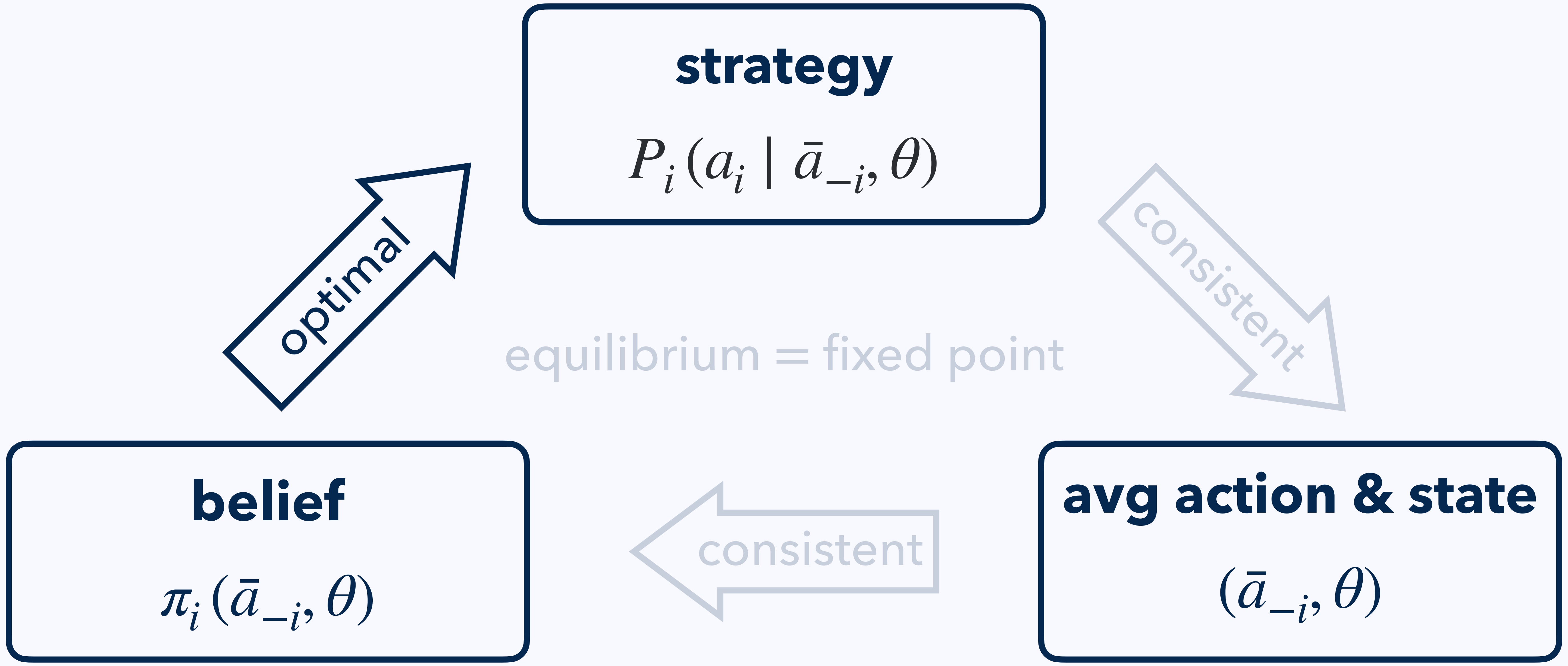


An **equilibrium** is a joint distribution $P_n^*(a_1, \ldots, a_n, \theta)$ such that:

- optimality: conditional distribution $P_n^*(a_i \mid \bar{a}_{-i}, \theta)$ = agent i's best response
- consistency: marginal distribution of state θ = prior π

focus on symmetric equilibria

Equilibrium Denti '23 Hébert–La'O '23

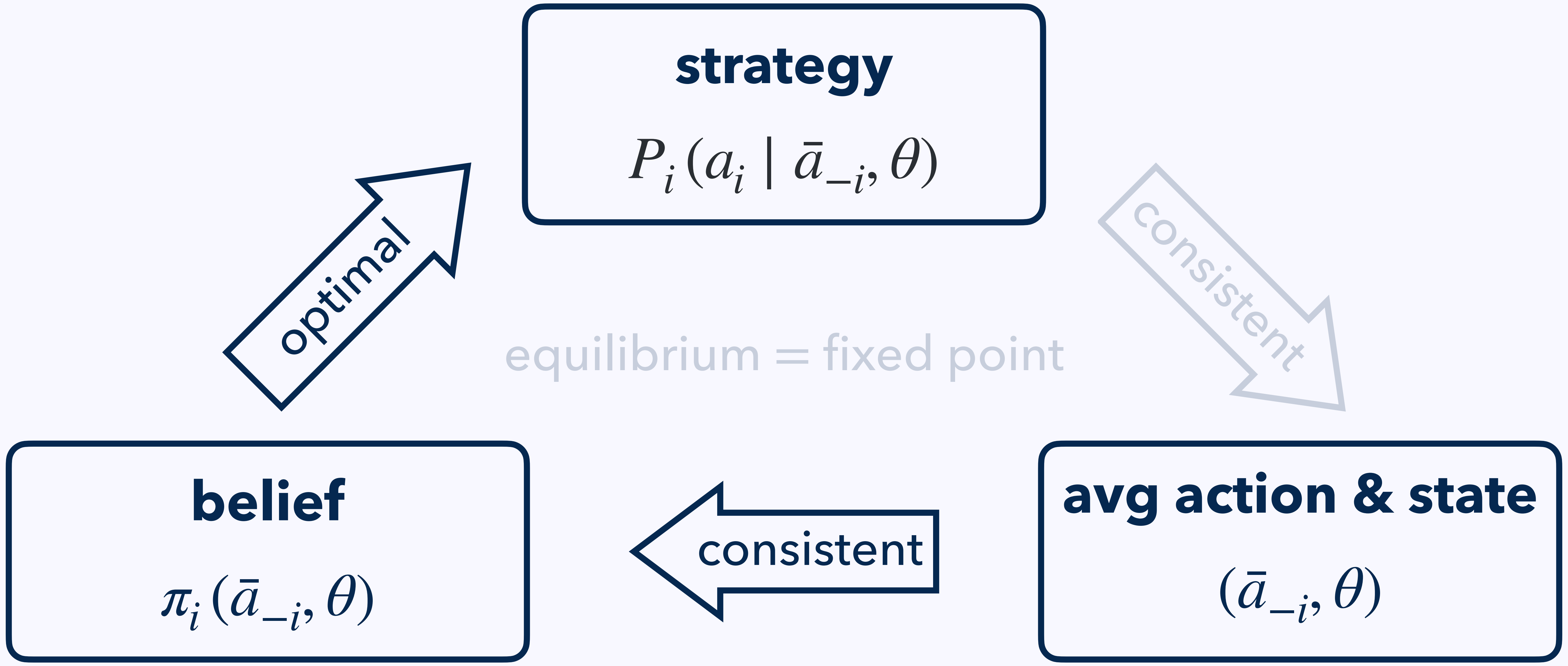


An **equilibrium** is a joint distribution $P_n^*(a_1, \ldots, a_n, \theta)$ such that:

- optimality: conditional distribution $P_n^*(a_i \mid \bar{a}_{-i}, \theta)$ = agent i's best response
- consistency: marginal distribution of state θ = prior π

focus on symmetric equilibria

Equilibrium Denti '23 Hébert–La'O '23

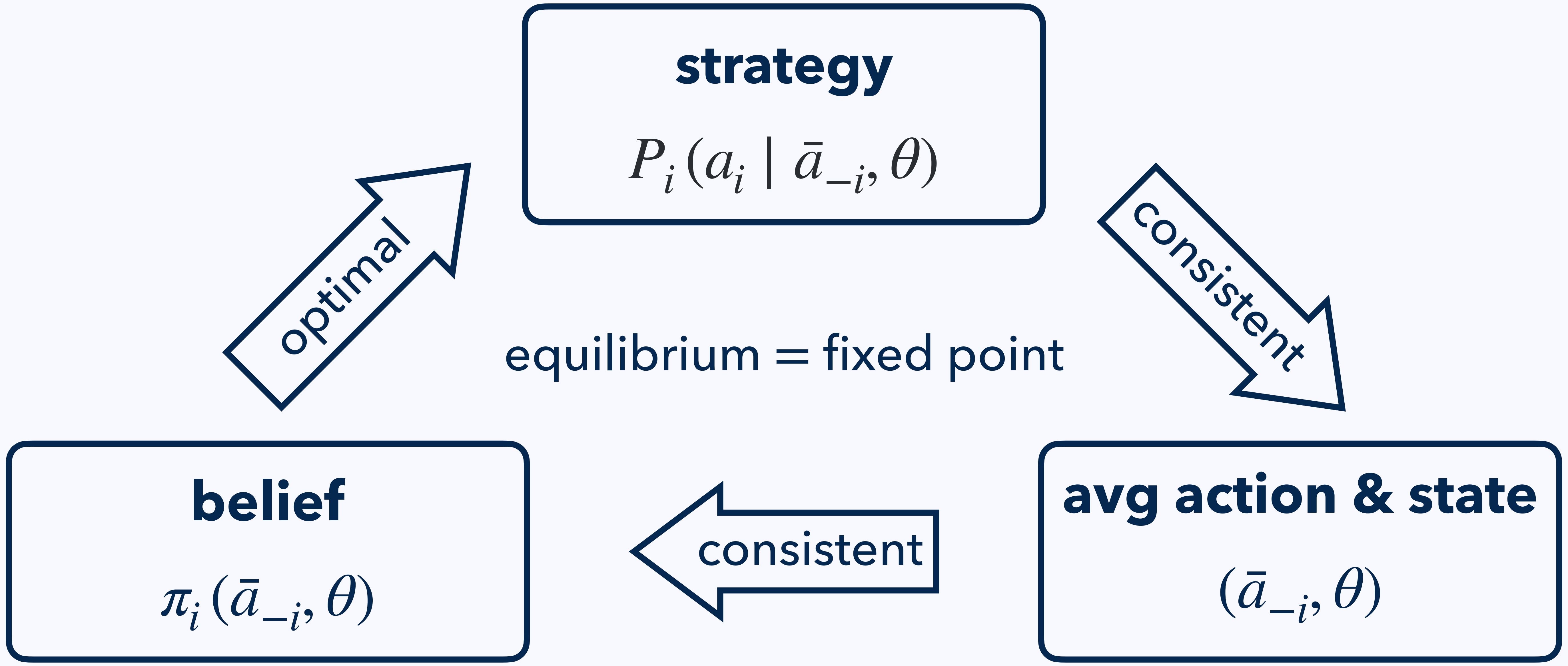


An **equilibrium** is a joint distribution $P_n^*(a_1, \ldots, a_n, \theta)$ such that:

- optimality: conditional distribution $P_n^*(a_i \mid \bar{a}_{-i}, \theta)$ = agent i's best response
- consistency: marginal distribution of state θ = prior π

focus on symmetric equilibria

Equilibrium Denti '23 Hébert–La'O '23



An **equilibrium** is a joint distribution $P_n^*(a_1, \ldots, a_n, \theta)$ such that:

- optimality: conditional distribution $P_n^*(a_i \mid \bar{a}_{-i}, \theta) =$ agent i's best response
- consistency: marginal distribution of state $\theta =$ prior π

focus on symmetric equilibria

Interpretations Denti '23 Hébert–La'O '23

A blend of two equilibrium concepts

- **Bayesian Nash Equilibrium**
  - agents optimize under uncertainty
- **Rational Expectations Equilibrium**
  - agents condition their behavior upon an endogenous average action

Grossman–Stiglitz (1980) etc.

- agents condition their behavior upon an endogenous average action (e.g., prices)

Every Equilibrium Is Biased-Logit Matějka–McKay '15 Denti '23

q_n^* is the unconditional action distribution

$$q_n^*(a_i) = \sum_{a_{-i}, \theta} P_n^*(a_i, a_{-i}, \theta)$$

$$P_n^*(a \mid \theta) = \frac{\exp\left[v_n(a, \theta)\right] \prod_{i=1}^{n} q_n^*(a_i)}{\text{normalizing constant}}$$

more

Every Equilibrium Is Biased-Logit

Matějka–McKay '15 Denti '23

q_n^* is the unconditional action distribution

$$q_n^*(a_i) = \sum_{a_{-i}, \theta} P_n^*(a_i, a_{-i}, \theta)$$

$$P_n^*(a \mid \theta) \quad \propto \quad \exp\left[v_n(a, \theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

more

Aggregate Action Distribution

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \equiv \sum_{a:\bar{a}_n \approx x} P_n^*(a \mid \theta)$$

$$\propto \sum_{a:\bar{a}_n \approx x} \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

$v_n(a,\theta)$ depends on average $\bar{a}_n \approx x$

$v_n(x,\theta)$ is continuous

$$\approx \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

potential reward vs **deviation penalty**

Aggregate Action Distribution

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \equiv \sum_{a:\bar{a}_n \approx x} P_n^*(a \mid \theta)$$

$$\propto \sum_{a:\bar{a}_n \approx x} \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

$v_n(a,\theta)$ depends on average $\bar{a}_n \approx x$

$v_n(x,\theta)$ is continuous

$$\approx \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

potential reward vs **deviation penalty**

Aggregate Action Distribution

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \equiv \sum_{a:\bar{a}_n \approx x} P_n^*(a \mid \theta)$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \propto \sum_{a:\bar{a}_n \approx x} \exp\left[v_n(a, \theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

$v_n(a, \theta)$ depends on average $\bar{a}_n \approx x$

$v_n(x, \theta)$ is continuous

$$\approx \exp\left[v_n(x, \theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

potential reward vs deviation penalty

Aggregate Action Distribution

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \equiv \sum_{a:\bar{a}_n \approx x} P_n^*(a \mid \theta)$$

$$\propto \sum_{a:\bar{a}_n \approx x} \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

$$\approx \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

$v_n(a,\theta)$ depends on average $\bar{a}_n \approx x$

$v_n(x,\theta)$ is continuous

potential reward vs **deviation penalty**

$$\exp\left[v_n(x,\theta)\right] = \exp\left[n \cdot \frac{v_n(x,\theta)}{n}\right] \approx \exp\left[n\, f(x,\theta)\right]$$

the **limit potential**:

$$\frac{v_n(x,\theta)}{n} \xrightarrow{n\to\infty} f(x,\theta) \equiv \frac{x^2}{2} - \theta\, x$$

$$\exp\left[v_n(x,\theta)\right] = \exp\left[n \cdot \frac{v_n(x,\theta)}{n}\right] \approx \exp\left[n\, f(x,\theta)\right]$$

the **limit potential**:

$$\frac{v_n(x,\theta)}{n} \xrightarrow{n\to\infty} f(x,\theta) \equiv \frac{x^2}{2} - \theta x$$

$$\sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i) = \sum_{k\,:\,k/n \approx x} \binom{n}{k} \left(q_n^*(1)\right)^k \left(q_n^*(0)\right)^{n-k} \approx \exp\left[-n\,\mathrm{KL}\left(x \,\|\, q^*\right)\right]$$

Kullback-Leibler Divergence (KLD)

$$\mathrm{KL}\left(x \,\|\, q^*\right) \equiv x \log \frac{x}{q^*} + (1-x) \log \frac{1-x}{1-q^*}$$

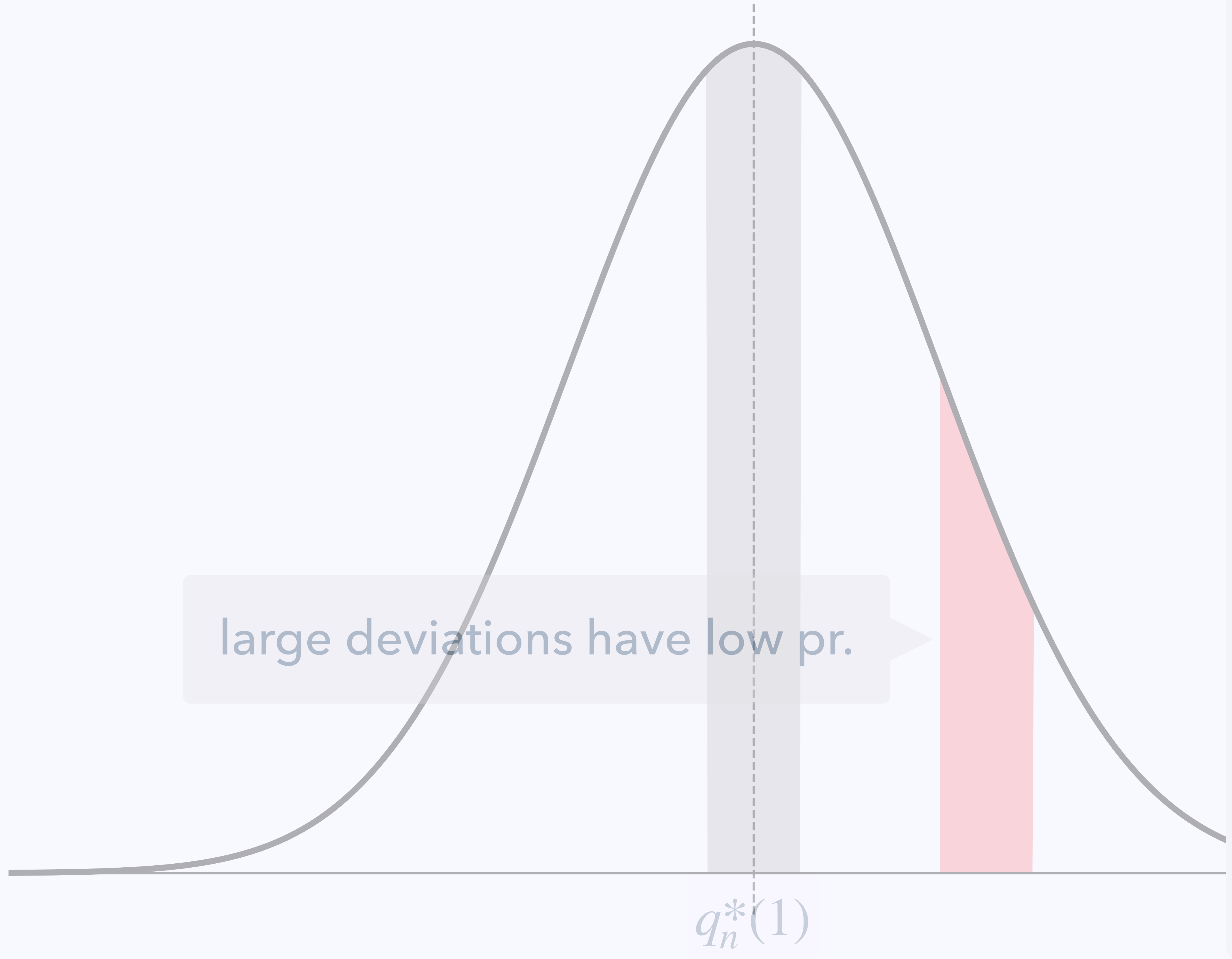
- $q^* \equiv \lim_n q_n^*(1)$
- $\mathrm{KL}\left(x \,\|\, q^*\right) \geq 0$ (with "=" $\Leftrightarrow x = q^*$)

Large Deviations Theory

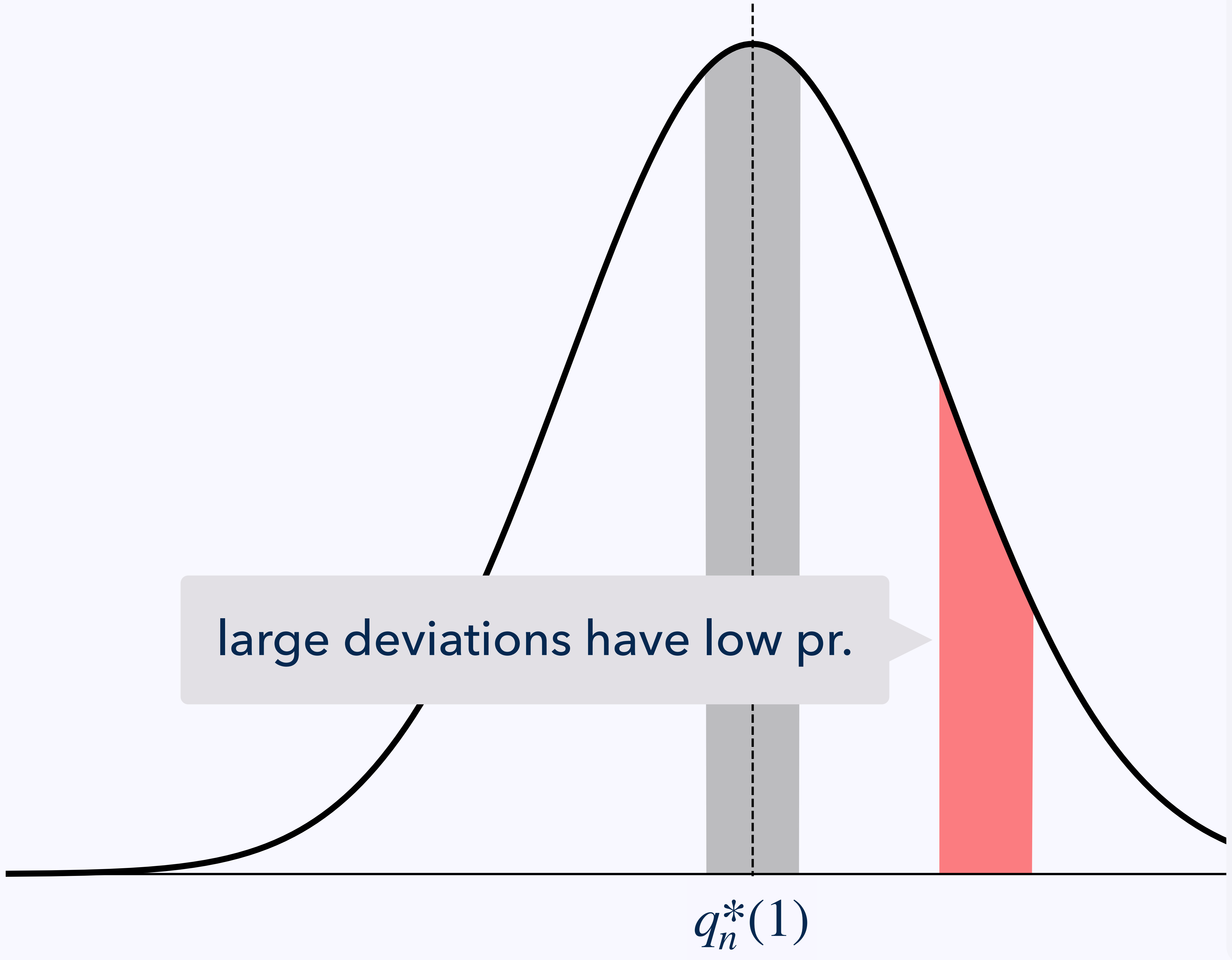
- how fast large deviations prob. decay

$$\Pr\left(\text{large deviations event}\right) \approx \exp\left[-n \cdot \textbf{convergence rate}\right]$$

- **non-iid** case but **KLD rate**



$$\sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i) = \sum_{k\,:\,k/n \approx x} \binom{n}{k} \left(q_n^*(1)\right)^k \left(q_n^*(0)\right)^{n-k} \approx \exp\left[-n\,\mathrm{KL}\left(x \,\|\, q^*\right)\right]$$



Kullback-Leibler Divergence (KLD)

$$\mathrm{KL}\left(x \,\|\, q^*\right) \equiv x \log \frac{x}{q^*} + (1-x) \log \frac{1-x}{1-q^*}$$

- $q^* \equiv \lim_n q_n^*(1)$
- $\mathrm{KL}\left(x \,\|\, q^*\right) \geq 0$ (with "=" $\Leftrightarrow x = q^*$)

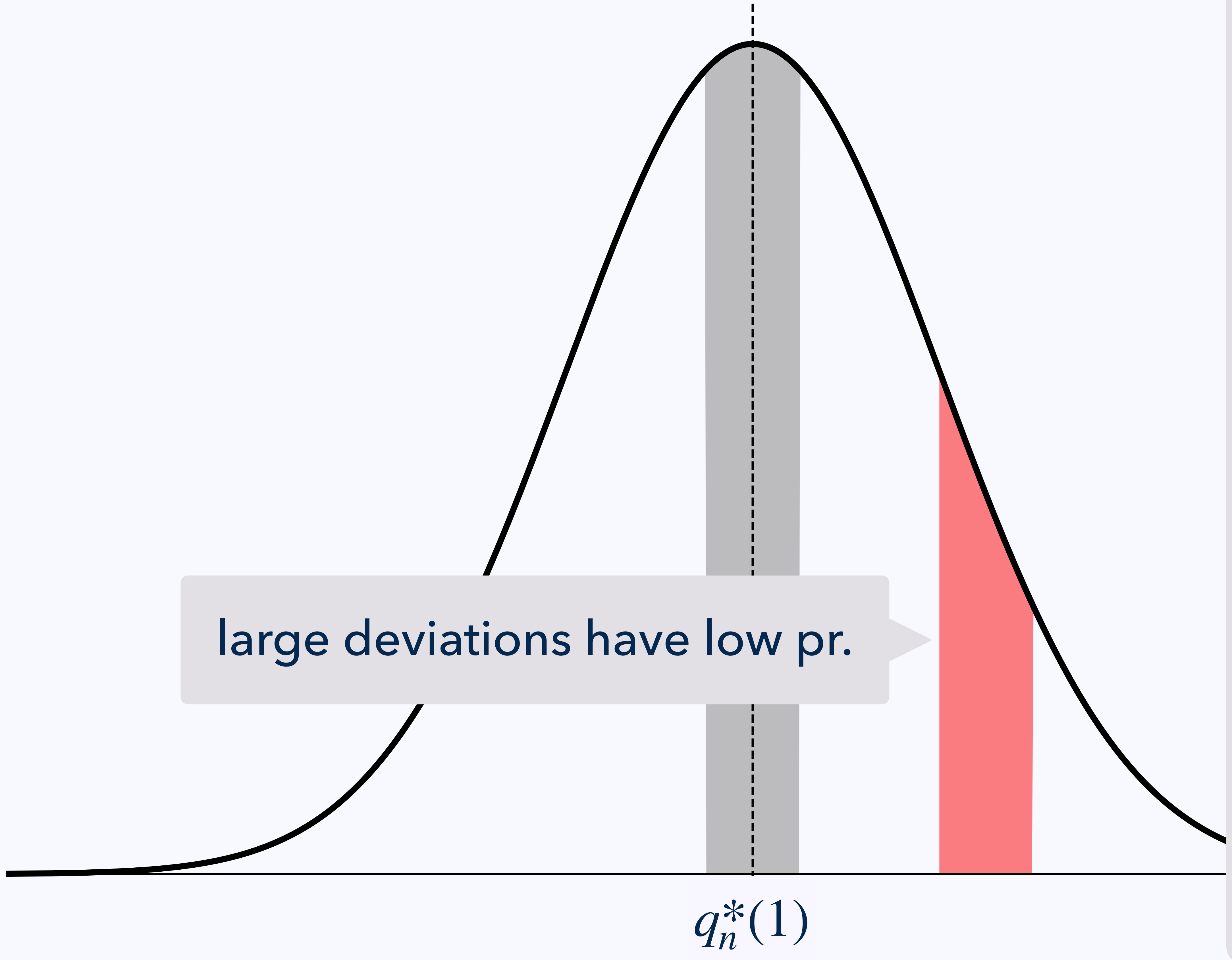
Large Deviations Theory

- how fast large deviations prob. decay

$$\Pr\left(\text{large deviations event}\right) \approx \exp\left[-n \cdot \textbf{convergence rate}\right]$$

- **non-iid** case but **KLD rate**

$$\sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i) = \sum_{k\,:\,k/n \approx x} \binom{n}{k} \left(q_n^*(1)\right)^k \left(q_n^*(0)\right)^{n-k} \approx \exp\left[-n\,\mathrm{KL}\left(x \,\|\, q^*\right)\right]$$



Kullback-Leibler Divergence (KLD)

$$\mathrm{KL}\left(x \,\|\, q^*\right) \equiv x \log \frac{x}{q^*} + (1-x) \log \frac{1-x}{1-q^*}$$

- $q^* \equiv \lim_n q_n^*(1)$
- $\mathrm{KL}\left(x \,\|\, q^*\right) \geq 0$ (with "=" $\Leftrightarrow x = q^*$)

Large Deviations Theory

- how fast large deviations prob. decay

$$\mathrm{Pr}\left(\text{large deviations event}\right) \approx \exp\left[-n \cdot \textbf{convergence rate}\right]$$

- **non-iid** case but **KLD rate**

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \propto \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

blue $\approx \exp\left[n f(x,\theta)\right]$

red $\approx \exp\left[-n\,\mathrm{KL}(x \,\|\, q^*)\right]$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \propto \exp\left[n\left[f(x,\theta) - \mathrm{KL}(x \,\|\, q^*)\right]\right]$$

KLD-regularized potential $F(x, q^*, \theta)$

- trade-off: potential reward vs deviation penalty

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \;\propto\; \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

$$\text{blue} \approx \exp\left[n\, f\left(x,\theta\right)\right]$$

$$\text{red} \approx \exp\left[-\,n\, \mathrm{KL}\left(x \,\|\, q^*\right)\right]$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \;\propto\; \exp\left[n\left[f\left(x,\theta\right) - \mathrm{KL}\left(x \,\|\, q^*\right)\right]\right]$$

KLD-regularized potential $F\left(x, q^*, \theta\right)$

- trade-off: potential reward vs deviation penalty

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \;\propto\; \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

$$\text{blue} \approx \exp\left[n\, f(x,\theta)\right]$$

$$\text{red} \approx \exp\left[-\, n\, \mathrm{KL}\left(x \,\|\, q^*\right)\right]$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \;\propto\; \exp\left[n\left[f(x,\theta) - \mathrm{KL}\left(x \,\|\, q^*\right)\right]\right]$$

KLD-regularized potential $F\left(x, q^*, \theta\right)$

- trade-off: potential reward vs deviation penalty

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \quad \propto \quad \exp\left[v_n(x,\theta)\right] \sum_{a:\bar{a}_n \approx x} \prod_{i=1}^{n} q_n^*(a_i)$$

$$\text{blue} \approx \exp\left[n\, f(x,\theta)\right]$$

$$\text{red} \approx \exp\left[-n\, \mathrm{KL}\left(x \,\|\, q^*\right)\right]$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \quad \propto \quad \exp\left[n\left[f(x,\theta) - \mathrm{KL}\left(x \,\|\, q^*\right)\right]\right]$$

KLD-regularized potential $F\left(x, q^*, \theta\right)$

- trade-off: potential reward vs deviation penalty

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \quad \propto \quad \exp\left[n\, F\left(x, q^*, \theta\right)\right]$$

KLD-regularized potential

$$F\left(x, q^*, \theta\right) \equiv f\left(x, \theta\right) - \mathrm{KL}\left(x \,\|\, q^*\right)$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \quad \lessapprox \quad \frac{\exp\left[n\, F\left(x, q^*, \theta\right)\right]}{\exp\left[n \max_z F\left(z, q^*, \theta\right)\right]}$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \quad = \quad \frac{\exp\left[n\, F\left(x, q^*, \theta\right)\right]}{\exp\left[n\, F\left(x^*(\theta), q^*, \theta\right)\right]}$$

KLD-regularized potential maximizer

$$x^*(\theta) = \arg\max_z F\left(z, q^*, \theta\right)$$

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \approx \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\text{normalizing constant}}$$

KLD-regularized potential

$$F\left(x, q^*, \theta\right) \equiv f\left(x, \theta\right) - \mathrm{KL}\left(x \,\|\, q^*\right)$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \lessapprox \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n \max_z F\left(z, q^*, \theta\right)\right]}$$

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) = \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n\,F\left(x^*(\theta), q^*, \theta\right)\right]}$$

KLD-regularized potential maximizer

$$x^*(\theta) = \arg\max_z F\left(z, q^*, \theta\right)$$

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \approx \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\text{normalizing constant}}$$

KLD-regularized potential

$$F\left(x, q^*, \theta\right) \equiv f\left(x, \theta\right) - \mathrm{KL}\left(x \parallel q^*\right)$$

$$\lessapprox \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n \max_z F\left(z, q^*, \theta\right)\right]}$$

$$= \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n\,F\left(x^*(\theta), q^*, \theta\right)\right]}$$

KLD-regularized potential maximizer

$$x^*(\theta) = \arg\max_z F\left(z, q^*, \theta\right)$$

video

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \approx \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\text{normalizing constant}}$$

KLD-regularized potential

$$F\left(x, q^*, \theta\right) \equiv f\left(x, \theta\right) - \mathrm{KL}\left(x \,\|\, q^*\right)$$

$$\lessapprox \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n \max_z F\left(z, q^*, \theta\right)\right]}$$

$$= \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n\,F\left(x^*(\theta), q^*, \theta\right)\right]}$$

KLD-regularized potential maximizer

$$x^*(\theta) = \arg\max_z F\left(z, q^*, \theta\right)$$

video

KLD-Regularized Potential

$$\Pr\left(\bar{a}_n \approx x \mid \theta\right) \approx \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\text{normalizing constant}}$$

KLD-regularized potential

$$F\left(x, q^*, \theta\right) \equiv f\left(x, \theta\right) - \mathrm{KL}\left(x \,\|\, q^*\right)$$

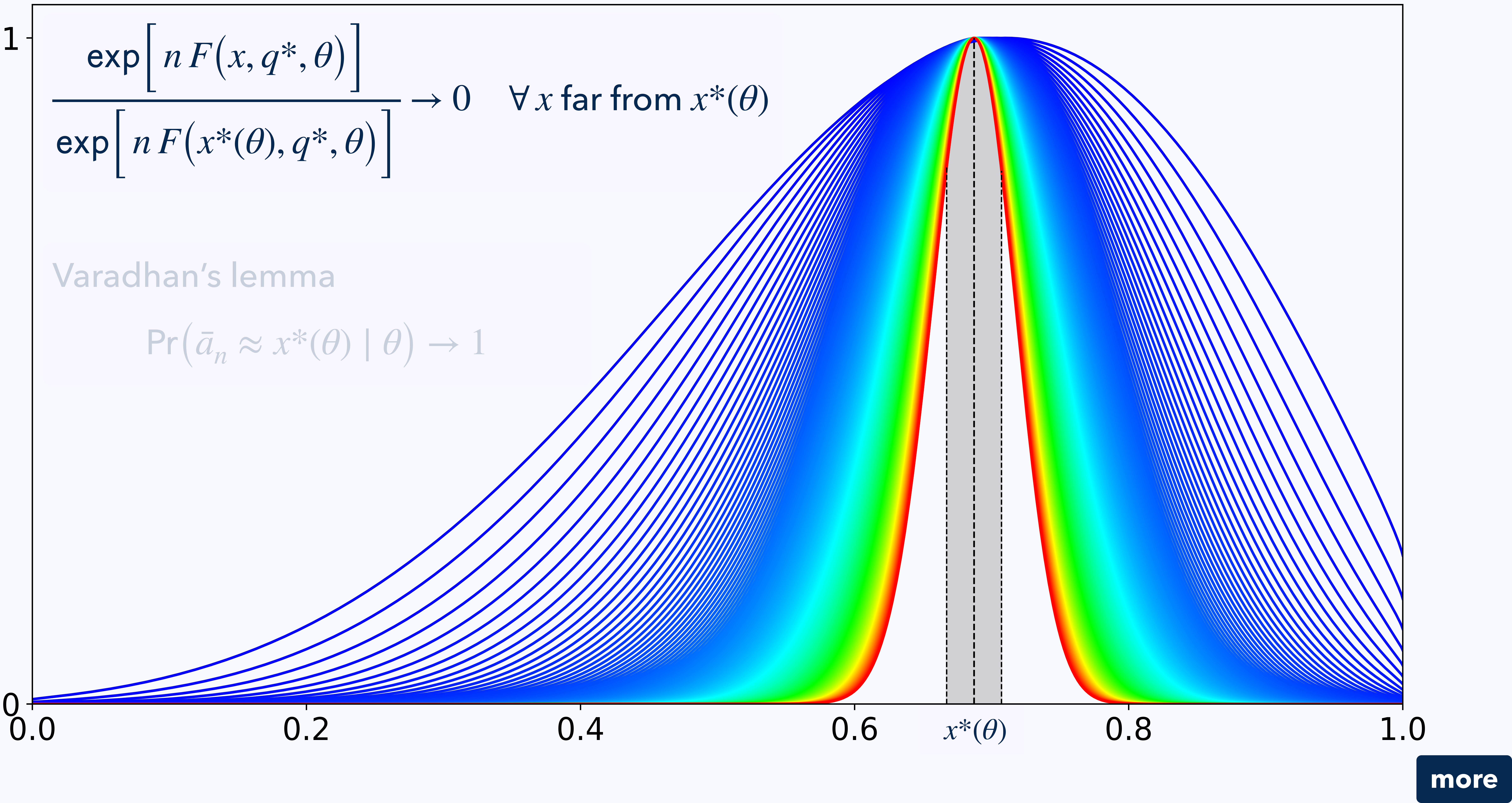
$$\lessapprox \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n \max_z F\left(z, q^*, \theta\right)\right]}$$

$$= \frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n\,F\left(x^*(\theta), q^*, \theta\right)\right]}$$

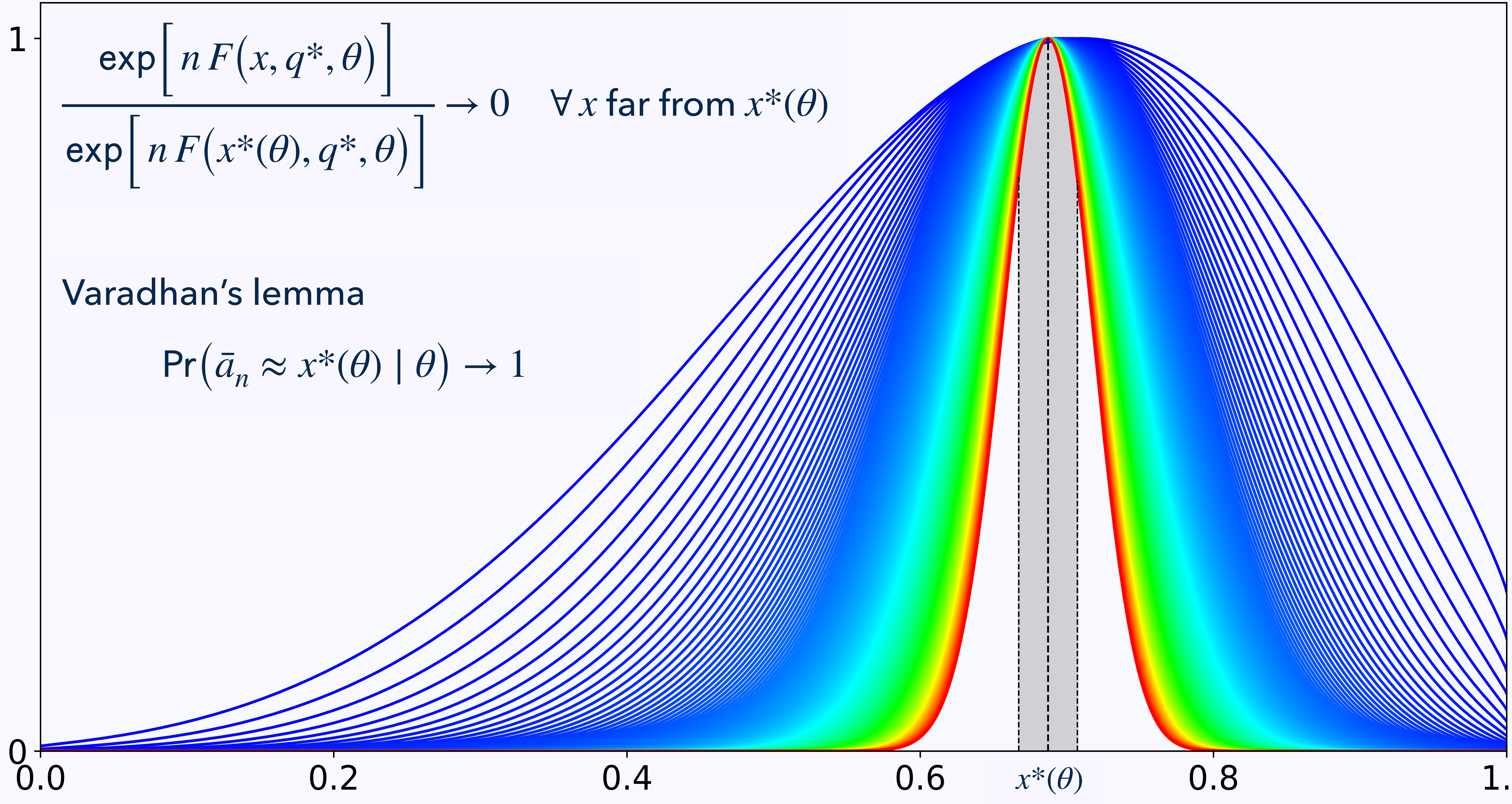
KLD-regularized potential maximizer

$$x^*(\theta) = \arg\max_z F\left(z, q^*, \theta\right)$$

video



$$\frac{\exp\left[n\,F\big(x, q^*, \theta\big)\right]}{\exp\left[n\,F\big(x^*(\theta), q^*, \theta\big)\right]} \to 0 \quad \forall\, x \text{ far from } x^*(\theta)$$
Varadhan's lemma
$\Pr\big(\bar{a}_n \approx x^*(\theta) \mid \theta\big) \to 1$
1
0
0.0
0.2
0.4
0.6
$x^*(\theta)$
0.8
1.0
more



$$\frac{\exp\left[n\,F\left(x, q^*, \theta\right)\right]}{\exp\left[n\,F\left(x^*(\theta), q^*, \theta\right)\right]} \to 0 \quad \forall\, x \text{ far from } x^*(\theta)$$
Varadhan's lemma
$$\Pr\left(\bar{a}_n \approx x^*(\theta) \mid \theta\right) \to 1$$
1
0
0.0
0.2
0.4
0.6
$x^*(\theta)$
0.8
1.0

more

Main Result: Equilibrium = Representative Agent

As #agents $n \to \infty$, the equilibrium is as if a **representative agent** were solving

- the KLD-regularized potential

$$x^*(\theta) \in \arg\max_x F(x, q^*, \theta)$$

  - $q^* = \lim_n q_n^*$ by definition

KLD-regularized potential

$$F(x, q^*, \theta) \equiv f(x, \theta) - \text{KL}(x \,\|\, q^*)$$

- the **expected limit potential – entropy costs**

$$(x^*, q^*) \in \arg\max_{x,q} \sum_\theta \pi(\theta) f(x(\theta), \theta) - \text{entropy costs}$$

similar to, but different from, individual problems

- **individual:** $f(x, \theta)$ is **linear** in x
- **representative agent:** $f(x, \theta)$ is **not linear** in x

Main Result: Equilibrium = Representative Agent

As #agents $n \to \infty$, the equilibrium is as if a **representative agent** were solving

- the KLD-regularized potential

$$x^*(\theta) \in \arg\max_x F(x, q^*, \theta)$$

KLD-regularized potential

$$F(x, q^*, \theta) \equiv f(x, \theta) - \mathrm{KL}(x \,\|\, q^*)$$

  - $q^* = \lim_n q_n^*$ by definition

- the **expected limit potential – entropy costs**

$$(x^*, q^*) \in \arg\max_{x,q} \sum_\theta \pi(\theta) f(x(\theta), \theta) - \text{entropy costs}$$

similar to, but different from, individual problems

- **individual:** $f(x, \theta)$ is **linear** in x
- **representative agent:** $f(x, \theta)$ is **not linear** in x

Main Result: Equilibrium = Representative Agent

As #agents $n \to \infty$, the equilibrium is as if a **representative agent** were solving

- the KLD-regularized potential

$$x^*(\theta) \in \arg\max_x F(x, q^*, \theta)$$

  - $q^* = \lim_n q_n^*$ by definition

KLD-regularized potential

$$F(x, q^*, \theta) \equiv f(x, \theta) - \mathrm{KL}(x \,\|\, q^*)$$

- the **expected limit potential – entropy costs**

$$(x^*, q^*) \in \arg\max_{x,q} \sum_\theta \pi(\theta) f(x(\theta), \theta) - \text{entropy costs}$$

similar to, but different from, individual problems

- **individual:** $f(x, \theta)$ is **linear** in x
- **representative agent:** $f(x, \theta)$ is **not linear** in x

Main Result: Equilibrium = Representative Agent

As #agents $n \to \infty$, the equilibrium is as if a **representative agent** were solving

- the KLD-regularized potential

$$x^*(\theta) \in \arg\max_x F(x, q^*, \theta)$$

KLD-regularized potential

$$F(x, q^*, \theta) \equiv f(x, \theta) - \mathrm{KL}(x \,\|\, q^*)$$

  - $q^* = \lim_n q_n^*$ by definition

- the **expected limit potential – entropy costs**

$$(x^*, q^*) \in \arg\max_{x,q} \sum_\theta \pi(\theta) f(x(\theta), \theta) - \text{entropy costs}$$

similar to, but different from, individual problems

- **individual:** $f(x, \theta)$ is **linear** in x
- **representative agent:** $f(x, \theta)$ is **not linear** in x

Main Result: Endogenous Volatility

As #agents $n \to \infty$, the equilibrium is such that

- for any closed $C = [\underline{c}, \overline{c}]$ with $\underline{c} < \overline{c}$

$$x^*(\theta) \notin C \implies \lim_{n\to\infty} \Pr\big(\bar{a}_n \in C \mid \theta\big) = 0$$ **exponentially fast**

- **endogenous volatility vanishes**

$$\lim_{n\to\infty} \mathrm{Var}\big(\bar{a}_n \mid \theta\big) = \lim_{n\to\infty} \mathrm{Cov}\big(a_i, a_j \mid \theta\big) = 0$$

more

Main Result: Endogenous Volatility

As #agents $n \to \infty$, the equilibrium is such that

- for any closed $C = [\underline{c}, \overline{c}]$ with $\underline{c} < \overline{c}$

$$x^*(\theta) \notin C \implies \lim_{n\to\infty} \text{Pr}\big(\bar{a}_n \in C \mid \theta\big) = 0$$ **exponentially fast**

- **endogenous volatility vanishes**

$$\lim_{n\to\infty} \text{Var}\big(\bar{a}_n \mid \theta\big) = \lim_{n\to\infty} \text{Cov}\big(a_i, a_j \mid \theta\big) = 0$$

more

Main Result: Endogenous Volatility

As #agents $n \to \infty$, the equilibrium is such that

- for any closed $C = [\underline{c}, \bar{c}]$ with $\underline{c} < \bar{c}$

$$x^*(\theta) \notin C \implies \lim_{n\to\infty} \Pr\left(\bar{a}_n \in C \mid \theta\right) = 0$$ **exponentially fast**

- **endogenous volatility vanishes**

$$\lim_{n\to\infty} \mathrm{Var}\left(\bar{a}_n \mid \theta\right) = \lim_{n\to\infty} \mathrm{Cov}\left(a_i, a_j \mid \theta\right) = 0$$

more

Correlated Information vs Independent Information

As #agents $n \to \infty$, the equilibrium is such that

correlated information $\to$ **independent** information

- **correlated** information
  - an exogenous state θ and an endogenous average $\bar{a}_n$
- **independent** information
  - an exogenous state θ

Why Not Law of Large Numbers?

As #agents $n \to \infty$, we **cannot** use the Law of Large Numbers

- correlation in finite-agent models

$$P_n^*(a \mid \theta) \quad \propto \quad \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

Infinite-agent model with a continuum Law of Large Numbers (e.g., Hébert–La'O '23)

- "$n = \infty$" equilibria $\neq$ "$n \to \infty$" equilibria

more

Takeaway

Model

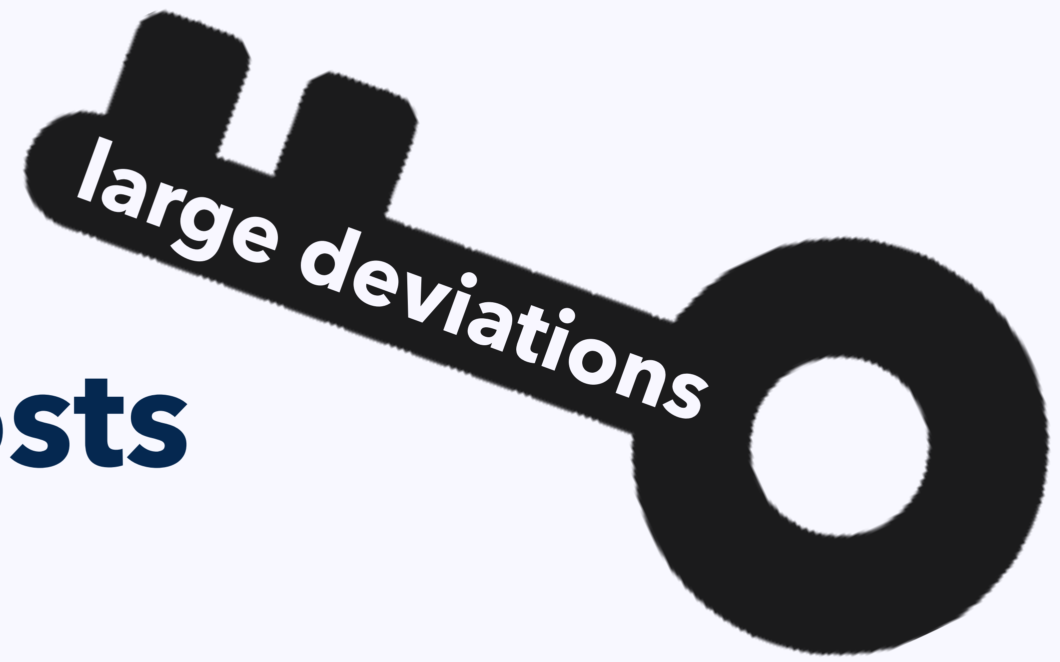
- all agents have a finite action space A
- $\exists$ potential v_n depends on a state θ and an empirical action distribution(a)
- $\exists$ limit potential $f \equiv \lim_{n\to\infty} \frac{v_n}{n}$

beauty contests, bank runs, political revolution, voting, etc.

Equilibrium

- representative agent's problem:

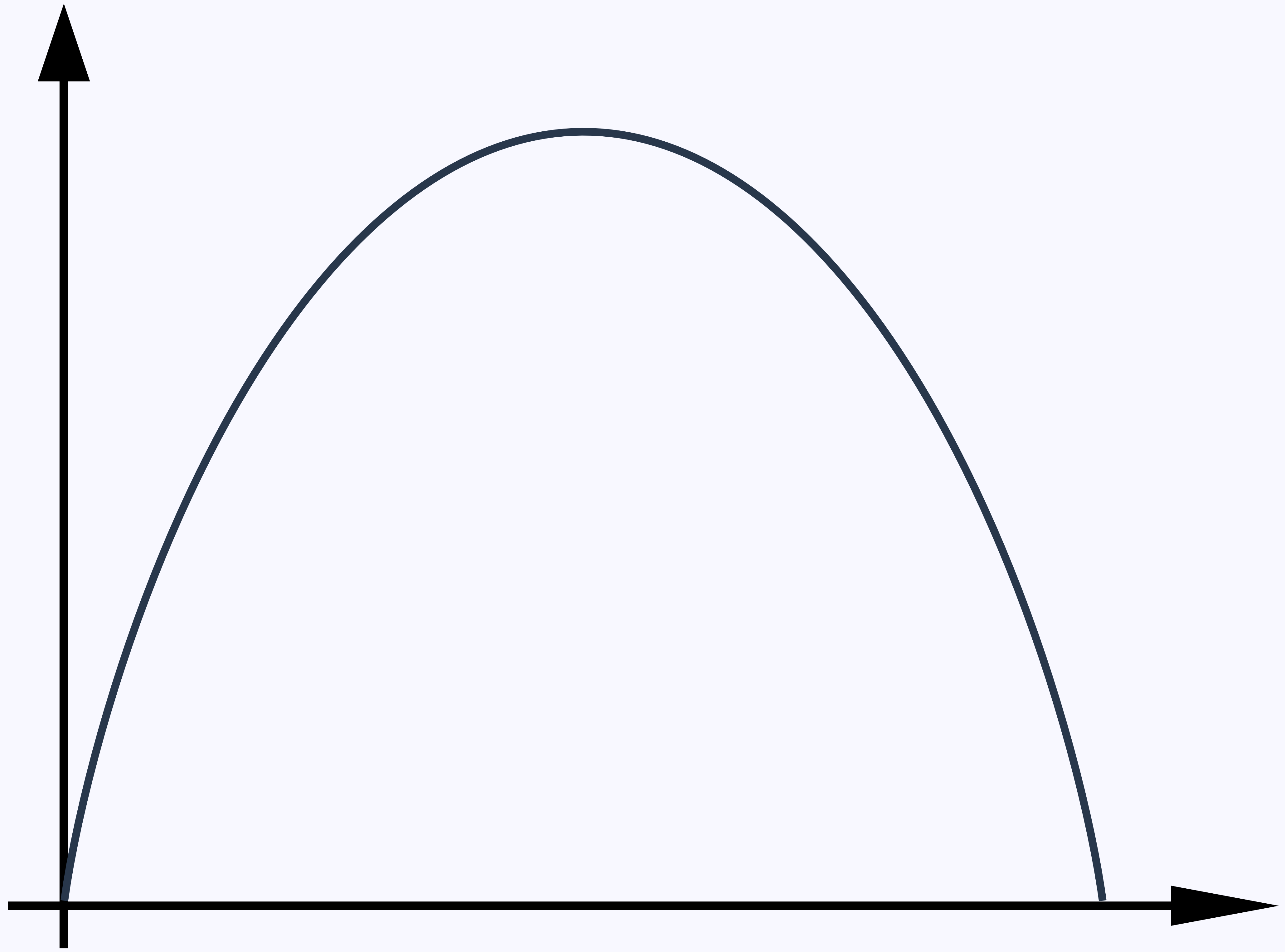
expected limit potential – entropy costs



- endogenous volatility **vanishes**

Appendix

Entropy

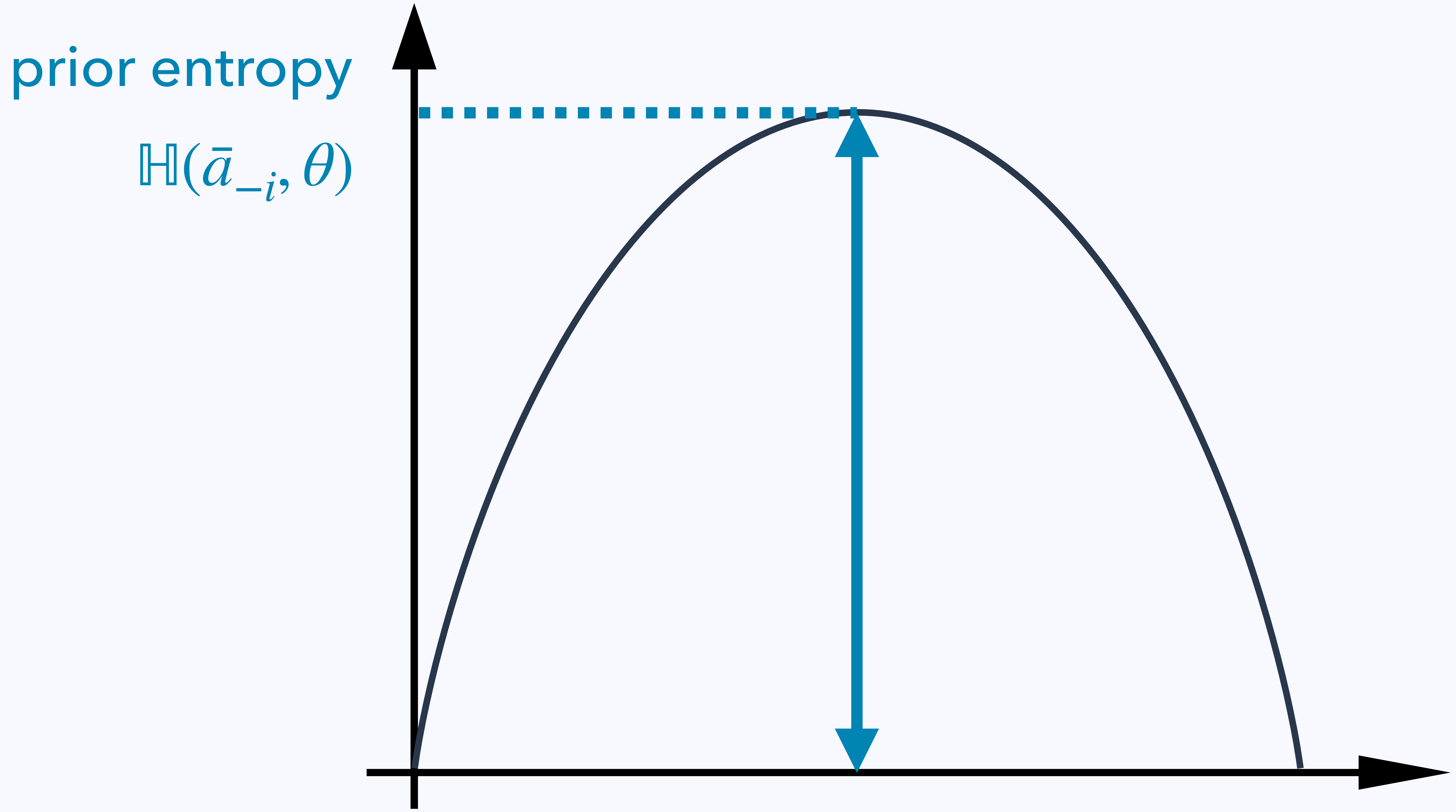


Entropy $\mathbb{H}(X)$ is a measure of uncertainty of a random variable X:

$$\mathbb{H}(X) = -\sum_x \Pr(x) \log(\Pr(x))$$

back

Entropy

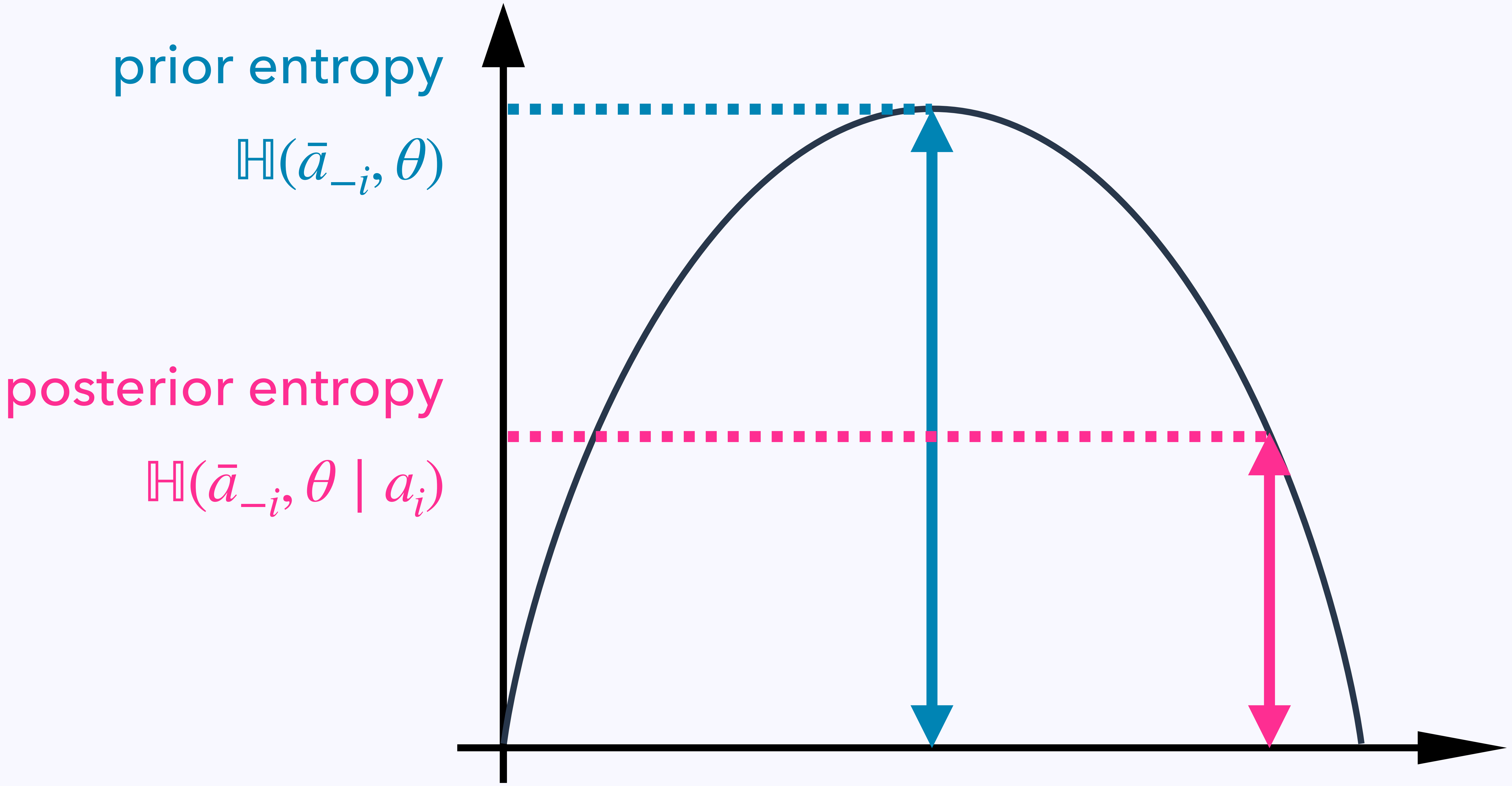


Entropy $\mathbb{H}(X)$ is a measure of uncertainty of a random variable X:

$$\mathbb{H}(X) = -\sum_x \Pr(x) \log(\Pr(x))$$

back

Entropy

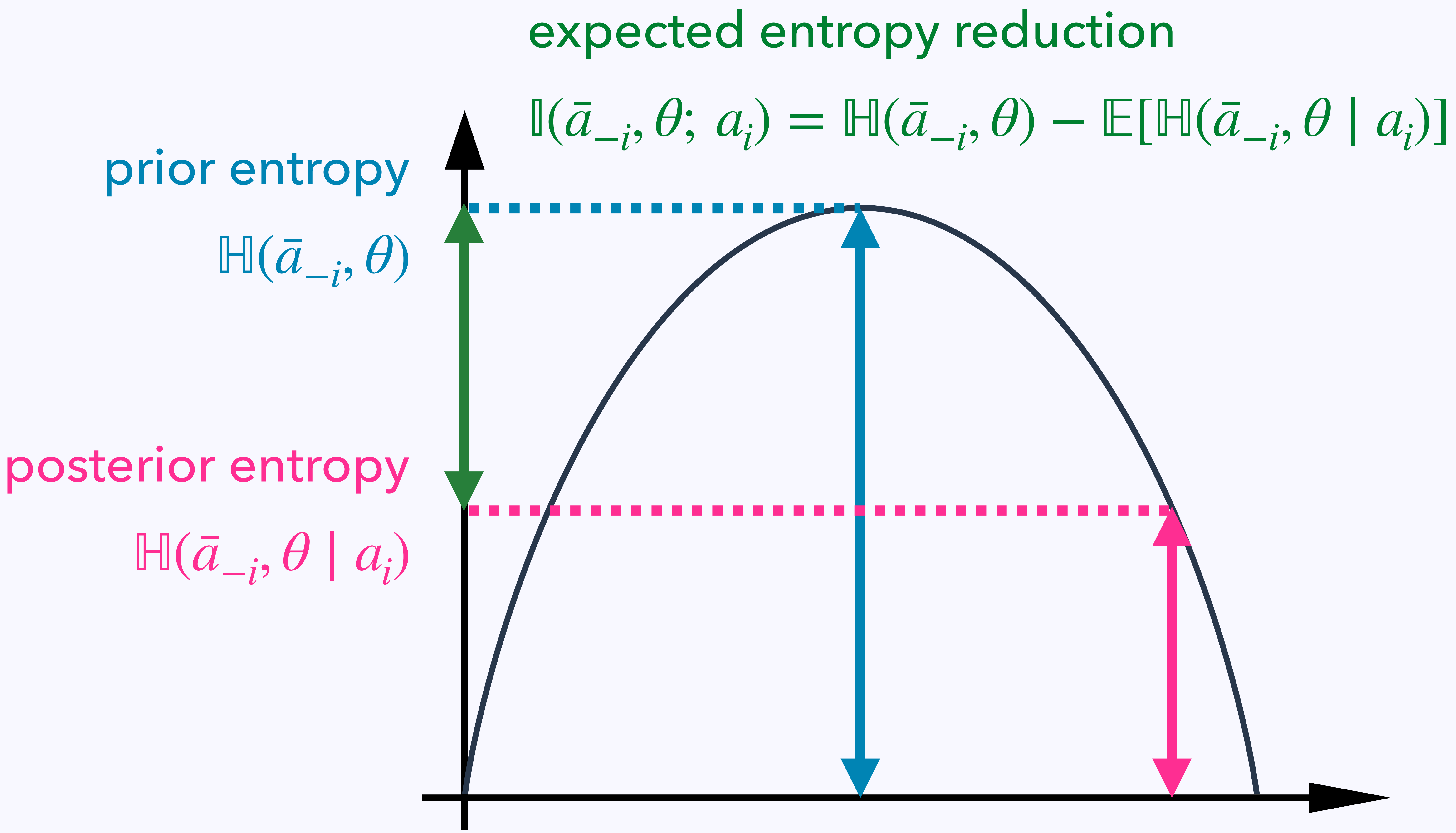


Entropy $\mathbb{H}(X)$ is a measure of uncertainty of a random variable X:

$$\mathbb{H}(X) = -\sum_x \text{Pr}(x) \log(\text{Pr}(x))$$

back

Entropy



Entropy $\mathbb{H}(X)$ is a measure of uncertainty of a random variable X:

$$\mathbb{H}(X) = -\sum_x \Pr(x) \log(\Pr(x))$$

back

Unconditional Action Distribution

$$P_n^*(a \mid \theta) \quad \propto \quad \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_n^*(a_i)$$

- q_n^* is the unconditional action distribution

$$q_n^*(a_i) = \sum_{a_{-i},\theta} P_n^*(a_i, a_{-i}, \theta)$$

- q_n^* is a symmetric pure NE of an auxiliary game (Denti '23)
  - each player chooses an action $q_i \in [0,1]$ and has a common payoff function:

$$U(q_1, \ldots, q_n) \equiv \sum_{\theta} \pi(\theta) \log \sum_{a} \exp\left[v_n(a,\theta)\right] \prod_{i=1}^{n} q_i(a_i)$$

  - e.g. $q_n^* \in \arg\max_q U(q, \ldots, q)$

back

Rate of Convergence

For any closed $C = [\underline{c}, \overline{c}]$ with $\underline{c} < \overline{c}$

$$x^*(\theta) \notin C \implies \lim_{n \to \infty} \Pr\left(\bar{a}_n \in C \mid \theta\right) = 0 \quad \textbf{exponentially fast}$$

The rate of convergence:

$$\lim_{n \to \infty} \frac{1}{n} \log \Pr\left(\bar{a}_n \in C \mid \theta\right) = \max_{x \in C} F\left(x, q^*, \theta\right) - \max_{x \in [0,1]} F\left(x, q^*, \theta\right)$$

For large n

$$\Pr\left(\bar{a}_n \in C \mid \theta\right) \approx \exp\left[n\left(\max_{x \in C} F\left(x, q^*, \theta\right) - \max_{x \in [0,1]} F\left(x, q^*, \theta\right)\right)\right]$$

Varadhan's Lemma

The family of probability measures $\{\mu_n\}_n$ satisfies the **Large Deviations Principle** if

$$\mu_n(E) \approx \text{exp}\left[-n \inf_{x \in E} I(x)\right]$$

- this function I gives the rate of convergence
  - KL divergence: $I(x) = \text{KL}(x \,\|\, q^*)$

Varadhan's Lemma:

$$\int \text{exp}\left[n\,\phi(x)\right]\mu_n(\text{d}x) \approx \text{exp}\left[n \sup_x\{\phi(x) - I(x)\}\right] \quad \forall \text{continuous and bounded } \phi$$

back

"$n = \infty$" Model vs "$n \to \infty$" Model

Example

- state $\theta = \pm 1$ with equally likely prior $\pi(\theta) = 1/2$
- agent i chooses action $a_i = \pm 1$

$$\text{payoff} \;\; u_i(a, \theta) = \begin{cases} +1 & \text{if } \operatorname{sgn}(\bar{a}_n) = \theta \\ -1 & \text{if } \operatorname{sgn}(\bar{a}_n) \neq \theta \end{cases}$$

"$n = \infty$" equilibrium $\neq$ "$n \to \infty$" equilibrium

- $n = \infty \implies$ each agent acquires **zero** info
- $n \to \infty \implies$ each agent acquires **some** info

back